Online Supplement for “Estimating smooth structural change in cointegrated system with mildly integrated regressors”

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The Online Supplement contains two additional Lemmas needed for the proofs of main theoretical results.

Lemma S.1. Consider the nearly integrated system as in Section 3. Let $\mathbf{u}_t = \left(\mathbf{u}_{0t}^\top, \mathbf{u}_{xt}^\top\right)^\top$ be defined as in (6). Under Assumptions 1, 2, and 3, we have, as $n \to \infty$:

(i) $n^{-1/2}\mathbf{X}_{\lfloor nr \rfloor} \Rightarrow \mathbf{K}_C(r)$, $0 < r \leq 1$;

(ii) $\sup_{t \in \overline{N}_{nz_0}(h)} \left\| \frac{\mathbf{X}_t - \mathbf{X}_{z_n}}{\sqrt{nh}} \right\| = O_p(1)$, where $\overline{N}_{nz_0}(h)$ is a set of integers in $N_{nz_0}(h) = [\lfloor (z_0 - h)n \rfloor, \lfloor (z_0 + h)n \rfloor]$ and $z_n = \lfloor n(z_0 - h) \rfloor$, $z_0 \in (0, 1)$.

Proof. (i) is almost identical to Lemma 3.1(a) in Phillips 1988. However, since we have a different formulation for $\mathbf{R}_n$, the proofs are recollected here for completeness. Let $\mathbf{S}_t^x = \sum_{j=1}^t \mathbf{u}_{xj}$ be the vector of partial sums. We construct

$$\mathbf{Z}_n(r) = n^{-1/2}\mathbf{S}_{\lfloor nr \rfloor}^x = n^{-1/2}\mathbf{S}_{j-1}^x, \quad \frac{j-1}{n} \leq r < \frac{j}{n},$$

where $j = 1, 2, \cdots, n$. As shown in Phillips 1987, $\mathbf{Z}_n(r)$ obeys a functional central limit theorem:

$$\mathbf{Z}_n(r) \Rightarrow \mathbf{B}_X(r), \tag{1}$$

where $\mathbf{B}_X(r)$ is a k-vector Brownian motion with covariance matrix $\mathbf{\Omega}_{xx}$.

Based on the recursion

$$\mathbf{X}_t = \sum_{j=1}^{t} \mathbf{R}_n^{t-j}\mathbf{u}_{xj} + \mathbf{R}_n^t \mathbf{X}_0,$$

we have

$$\begin{aligned}
n^{-1/2}\mathbf{X}_{\lfloor nr \rfloor} &= \sum_{j=1}^{\lfloor nr \rfloor} \left(\mathbf{I}_k + \frac{\mathbf{C}}{n}\right)^{\lfloor nr \rfloor - j} \int_{\frac{j-1}{n}}^{\frac{j}{n}} d\mathbf{Z}_n(s) + O_p(n^{-1/2}) \\
&= \sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left(\mathbf{I}_k + \frac{\mathbf{C}}{n}\right)^{\lfloor nr \rfloor - j} d\mathbf{Z}_n(s) + O_p(n^{-1/2}) \\
&= \sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} \left[\left(\mathbf{I}_k + \frac{\mathbf{C}}{n}\right)^{\lfloor nr \rfloor - j} - e^{(r-s)\mathbf{C}}\right] d\mathbf{Z}_n(s) \\
&\quad + \sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) + O_p(n^{-1/2}) \\
&\leq \mathbf{T}_n \sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} d\mathbf{Z}_n(s) + \sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) + O_p(n^{-1/2}),
\end{aligned} \tag{2}$$

where

$$\mathbf{T}_n := \max_{1 \leq j \leq n} \left\{ \sup_{\frac{j-1}{n} \leq s < \frac{j}{n}} \left\| \left(\mathbf{I}_k + \frac{\mathbf{C}}{n}\right)^{\lfloor nr \rfloor - j} - e^{(r-s)\mathbf{C}} \right\| \right\}.$$

Notice that

$$\begin{aligned}
\sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) &= \int_0^{\lfloor nr \rfloor / n} e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) \\
&= \int_0^r e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) + O_p(1/n) \\
&= \mathbf{Z}_n(r) + \mathbf{C}\int_0^r e^{(r-s)\mathbf{C}} \mathbf{Z}_n(s) ds + O_p(1/n).
\end{aligned}$$

By (1) and the continuous mapping theorem (C.M.T.), we have

$$\mathbf{Z}_n(r) + \mathbf{C}\int_0^r e^{(r-s)\mathbf{C}} \mathbf{Z}_n(s) ds \Rightarrow \mathbf{B}_X(r) + \mathbf{C}\int_0^r e^{(r-s)\mathbf{C}} \mathbf{B}_X(s) ds.$$

This implies that $\sum_{j=1}^{\lfloor nr \rfloor} \int_{\frac{j-1}{n}}^{\frac{j}{n}} e^{(r-s)\mathbf{C}} d\mathbf{Z}_n(s) \Rightarrow \mathbf{K}_C(r)$, where $\mathbf{K}_C(r) = \mathbf{B}_X(r) + \mathbf{C}\int_0^r e^{(r-s)\mathbf{C}} \mathbf{B}_X(s) ds$.

What remains is to show that $\mathbf{T}_n = o_p(1)$. Observe that both $\mathbf{I}_k + \frac{\mathbf{C}}{n}$ and $e^{(r-s)\mathbf{C}}$ are diagonal matrix. It is suffice to verify that, for $i = 1, 2, \cdots, k$,

$$\max_{1\le j\le n}\left\{\sup_{\frac{j-1}{n}\le s<\frac{j}{n}}\left|\left(1+\frac{c_i}{n}\right)^{\lfloor nr\rfloor-j} - e^{c_i(r-s)}\right|\right\} = o(1).$$

We follow closely the steps in the proofs of Lemma 2.2 in Chan and Wei 1987. Define $d_n^i = -\frac{n}{c_i}\log\left(1+\frac{c_i}{n}\right)$. Clearly, we have $d_n^i \to -1$ as $n \to \infty$. For $\frac{j-1}{n} \le s < \frac{j}{n}$, $j = 1, 2, \cdots, n$, observe that

$$\begin{aligned}\left|\left(1+\frac{c_i}{n}\right)^{\lfloor nr\rfloor-j} - e^{c_i(r-s)}\right| &= e^{c_i\left(r-\frac{j}{n}\right)}\left|e^{\left(d_n^i+1\right)c_i\left(\frac{j}{n}-r\right)} - e^{c_i\left(\frac{j}{n}-s\right)}\right| \\ &\le e^{|c_i|}\left\{\left|e^{\left(d_n^i+1\right)c_i\left(\frac{j}{n}-r\right)}-1\right| + \left|1-e^{c_i\left(\frac{j}{n}-s\right)}\right|\right\}.\end{aligned}$$

The proof is completed by noticing that

$$\max_{1\le j\le n}\left|(d_n^i+1)c_i(\frac{j}{n}-r)\right| \le |c_i|\left|d_n^i+1\right| \to 0,$$

and

$$\max_{1\le j\le n}\sup_{\frac{j-1}{n}\le s<\frac{j}{n}}\left|1-e^{c_i\left(\frac{j}{n}-s\right)}\right| \le \max_{1\le j\le n}\left\{e^{c_i/n}-1, 1-e^{c_i/n}\right\} \to 0,$$

as $n \to \infty$.

For (ii), we first notice that

$$\begin{aligned}\frac{1}{\sqrt{2\lfloor nh\rfloor}}\left(\mathbf{X}_t - \mathbf{X}_{z_n}\right) &= \frac{1}{\sqrt{2\lfloor nh\rfloor}}\sum_{j=z_n+1}^{t}\mathbf{R}_n^{t-j}\mathbf{u}_{xj} + \frac{1}{\sqrt{2\lfloor nh\rfloor}}\sum_{j=1}^{z_n}\left(\mathbf{R}_n^{t-j}-\mathbf{R}_n^{z_n-j}\right)\mathbf{u}_{xj} \\ &\quad + \frac{1}{\sqrt{2\lfloor nh\rfloor}}\left(\mathbf{R}_n^{t}-\mathbf{R}_n^{z_n}\right)\mathbf{X}_0.\end{aligned}$$

Define the localized partial-sum process

$$\mathbf{S}_n(r) := \frac{1}{\sqrt{2\lfloor nh\rfloor}}\sum_{j=z_n+1}^{z_n+\lfloor 2\lfloor nh\rfloor r\rfloor}\mathbf{R}_n^{z_n+\lfloor 2\lfloor nh\rfloor r\rfloor-j}\mathbf{u}_{xj}, \qquad r\in[0,1].$$

Then, for each $t \in \overline{N}_{nz_0}(h)$ we may write

$$\frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=z_n+1}^{t} \mathbf{R}_n^{t-j} \mathbf{u}_{xj} = \mathbf{S}_n(r_t), \qquad r_t := \frac{t - z_n}{2\lfloor nh \rfloor} \in [-1, 1].$$

Following the same arguments as in part (i), we have $\mathbf{S}_n(\cdot) \Rightarrow \mathbf{K}_C(\cdot)$. Since $\mathbf{K}_C(\cdot)$ has continuous sample paths on the compact set $[-1, 1]$, the mapping $f \mapsto \sup_{r \in [-1,1]} \|f(r)\|$ is continuous. Hence, by C.M.T.,

$$\sup_{t \in \overline{N}_{nz_0}(h)} \left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=z_n+1}^{t} \mathbf{R}_n^{t-j} \mathbf{u}_{xj} \right\| \Rightarrow \sup_{-1 \le r \le 1} \|\mathbf{K}_C(r)\|.$$

This implies that

$$\sup_{t \in \overline{N}_{nz_0}(h)} \left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=z_n+1}^{t} \mathbf{R}_n^{t-j} \mathbf{u}_{xj} \right\| = O_p(1).$$

In addition, using the relation

$$\mathbf{R}_n^{t-j} - \mathbf{R}_n^{z_n-j} = \mathbf{R}_n^{z_n-j}\big(\mathbf{R}_n^{t-z_n} - \mathbf{I}_k\big), \qquad 1 \le j \le z_n, \tag{3}$$

we have

$$\sup_{t \in \overline{N}_{nz_0}(h)} \max_{1 \le j \le z_n} \|\mathbf{R}_n^{t-j} - \mathbf{R}_n^{z_n-j}\| \le O(1) \sup_{|m| \le nh} \|\mathbf{R}_n^m - \mathbf{I}_k\| = O(h), \tag{4}$$

which follows from the fact that $\sup_{0 \le m \le n} \|\mathbf{R}_n^m\| = O(1)$ and $\sup_{|m| \le nh} \|\mathbf{R}_n^m - \mathbf{I}_k\| = O(h)$. Moreover, $\| \sum_{j=1}^{z_n} \mathbf{u}_{xj} \| = O_p(\sqrt{n})$, so $\left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=1}^{z_n} \mathbf{u}_{xj} \right\| = O_p(1/\sqrt{h})$. Therefore,

$$\left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=1}^{z_n} \left(\mathbf{R}_n^{t-j} - \mathbf{R}_n^{z_n-j}\right) \mathbf{u}_{xj} \right\| \le \max_{1 \le j \le z_n} \left\|\mathbf{R}_n^{t-j} - \mathbf{R}_n^{z_n-j}\right\| \left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \sum_{j=1}^{z_n} \mathbf{u}_{xj} \right\| = o_p(1),$$

uniformly over $t \in \overline{N}_{nz_0}(h)$. We also have

$$\left\| \frac{1}{\sqrt{2\lfloor nh \rfloor}} \left(\mathbf{R}_n^{t} - \mathbf{R}_n^{z_n}\right) \mathbf{X}_0 \right\| \le \frac{1}{\sqrt{2\lfloor nh \rfloor}} \left\|\mathbf{R}_n^{t} - \mathbf{R}_n^{z_n}\right\| \|\mathbf{X}_0\| = o_p(1)$$

This completes the proof of (ii).

□

Lemma S.2. Consider the mildly integrated system as in Section 4. Suppose that Assumptions 1, 2, and 3 are satisfied. Then, for any fixed $z_0 \in (0,1)$, we have

(i) $\frac{1}{n^{1+\alpha/2}h}\sum_{t=1}^{n}\left\|K\left(\frac{t-nz_0}{nh}\right)\left[(\mathbf{R}_n^{t-1}\mathbf{X}_0)\otimes\tilde{\boldsymbol{u}}_t\right]\right\| = o_p(1)$;

(ii) $\frac{1}{nh}\sum_{t=1}^{n}\left\|K\left(\frac{t-nz_0}{nh}\right)\left[(\mathbf{R}_n^{t-1}\mathbf{X}_0)\otimes\boldsymbol{\varepsilon}_t\right]\right\| = o_p(1)$.

Proof. First, let ρ_{ni} be the ith element in $\mathbf{R}_n$, where $i = 1, 2, \cdots, k$. Since

$$(\rho_{ni})^{t-1} = \exp\left\{(t-1)\log\left(1+\frac{c_i}{n^\alpha}\right)\right\} \leq \exp\left((c-1)\frac{c_i}{n^\alpha}\right) \equiv \exp\left(-|c_i|\frac{t-1}{n^\alpha}\right),$$

this implies that

$$\left\|\mathbf{R}_n^{t-1}\right\|_{sp} = \max_{1\leq i\leq k}\left|(\rho_{ni})^{t-1}\right| \leq \exp\left(-c^*\frac{t-1}{n^\alpha}\right),$$

where $c^* := \min_{1\leq i\leq k}\ |c_i|$.

Let $N_{nz_0}(h) = [\lfloor(z_0-h)n\rfloor, \lfloor(z_0+h)n\rfloor]$ and let $z_n = \lfloor z_0 n\rfloor$. We then have

$$\begin{aligned}\frac{1}{nh}\sum_{t=1}^{n}K\left(\frac{t-nz_0}{nh}\right)\exp\left(-c^*\frac{t-1}{n^\alpha}\right) &= \frac{1}{nh}\sum_{t\in N_{nz_0}(h)}K\left(\frac{t-nz_0}{nh}\right)\exp\left(-c^*\frac{t-1}{n^\alpha}\right)\\ &\leq \exp\left(-c^*\frac{\lfloor(z_0-h)n\rfloor-1}{n^\alpha}\right)\frac{1}{nh}\sum_{t\in N_{nz_0}(h)}K\left(\frac{t-nz_0}{nh}\right)\\ &\leq O(1)\exp\left(-c_1 n^{1-\alpha}\right) = o(1),\end{aligned}$$

for some constant $c_1 > 0$. Then, Cauchy-Schwarz inequality gives

$$\begin{aligned}&\mathbb{E}\left(\frac{1}{n^{1+\alpha/2}h}\sum_{t=1}^{n}\left\|K\left(\frac{t-nz_0}{nh}\right)\left[(\mathbf{R}_n^{t-1}\mathbf{X}_0)\otimes\tilde{\boldsymbol{u}}_t\right]\right\|\right)\\ &\leq \frac{1}{n^{1+\alpha/2}h}\sum_{t=1}^{n}K\left(\frac{t-nz_0}{nh}\right)\left\|\mathbf{R}_n^{t-1}\right\|_{sp}\left(\mathbb{E}\left\|\mathbf{X}_0\right\|^2\right)^{1/2}\left(\mathbb{E}\left\|\tilde{\boldsymbol{u}}_t\right\|^2\right)^{1/2}\\ &\leq \frac{O(1)}{n^{\alpha/2}}\left(\frac{1}{nh}\sum_{t=1}^{n}K\left(\frac{t-nz_0}{nh}\right)\exp\left(-c^*\frac{t-1}{n^\alpha}\right)\right) = o(1),\end{aligned}$$

which established (i). (ii) follows from the same argument as (i), and the details are omitted. □

References

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