The model is an extension of Bai and Ng (2006):

$$y_{r,t+h} = \alpha' G_t + \beta' F_{r,t} + \delta' W_{r,t} + \varepsilon_{r,t+h}, \tag{1}$$

$$X_{r,it} = \gamma'_{r,i} G_t + \lambda'_{r,i} F_{r,t} + e_{r,it}, \tag{2}$$

where $r = 1, 2, \cdots, R$, $i = 1, 2, \cdots, N_r$, $t = 1, 2, \cdots, T$. $F_t = (F_{1,t}, F_{2,t}, \cdots, F_{R,t})'$ is a $r_F \times 1$ vector collecting all local factors and $r_F = \sum_r r_{F,r}$.

Let $N = \max_{1 \leqslant r \leqslant R} N_r$. Asymptotic analyses are carried out as $\underline{m} \to \infty$, where $\underline{m} = \min(R, N, T)$.

We will follow Algorithms 2.1-2.3 in Jin et al. (2023) to estimate the global factors and local factors for which we denote as $\tilde{G}_t$ and $\tilde{F}_t$. Let $\hat{z}_t = (\tilde{G}'_t, \tilde{F}'_{r,t}, W'_{r,t})'$ be the collection of predictors, the OLS estimator of $\theta = (\alpha', \beta', \delta')'$ is given by

$$\hat{\theta}_T = \Big(\sum_{t=h+1}^{T} \hat{z}_t \hat{z}'_t \Big)^{-1} \Big(\sum_{t=h+1}^{T} \hat{z}_t y_{t+h} \Big). \tag{3}$$

The first goal of the paper is to study the asymptotic properties of (3), which also provides a valid theoretical justification of the empirical application considered in Ergemen (2022).

First, notice that

$$\begin{aligned} y_{r,t+h} &= \alpha' G_t + \beta' F_{r,t} + \delta' W_{r,t} + \varepsilon_{r,t+h} \\ &= \underbrace{\alpha' H_G^{-1} \tilde{G}_t + \beta' H_{F_r}^{-1} \tilde{F}_{r,t} + \delta' W_{r,t}} + \alpha' H_G^{-1}(H_G G_t - \tilde{G}_t) + \beta' H_{F_r}^{-1}(H_{F_r} F_{r,t} - \tilde{F}_{r,t}) + \varepsilon_{r,t+h} \\ &= \hat{z}'_t \theta + \varepsilon_{r,t+h} + \alpha' H_G^{-1}(H_G G_t - \tilde{G}_t) + \beta' H_{F_r}^{-1}(H_{F_r} F_{r,t} - \tilde{F}_{r,t}). \end{aligned} \tag{4}$$

Then, write

$$\begin{aligned} \sqrt{T}\left(\hat{\theta}_T - \theta\right) = \underbrace{\Big(\frac{1}{T}\sum_{t=h+1}^{T} \hat{z}_t \hat{z}'_t\Big)^{-1}}_{S_{\hat{z}\hat{z}'}} \times \Bigg\{ \underbrace{\frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} H_G G_t \\ H_{F_r} F_{r,t} \\ W_t \end{pmatrix} \varepsilon_{t+h}}_{S_{C\varepsilon}} + \underbrace{\frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} \tilde{G}_t - H_G G_t \\ \tilde{F}_{r,t} - H_{F_r} F_{r,t} \\ 0 \end{pmatrix} \varepsilon_{t+h}}_{\Delta_{C\varepsilon}} \\ - \underbrace{\frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \hat{z}_t \big(\tilde{G}_t - H_G G_t\big)' \big(H_G^{-1}\big)' \alpha}_{\Delta_{zG}} - \underbrace{\frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \hat{z}_t \big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' \big(H_{F_r}^{-1}\big)' \beta}_{\Delta_{zF}} \Bigg\}, \end{aligned} \tag{5}$$

where $\theta = (\alpha' H_G^{-1}, \beta' H_{F_r}^{-1}, \delta')'$.

For $S_{C\varepsilon}$, define $\Phi_0 = \text{diag}\big(\operatorname{plim} H_G, \operatorname{plim} H_{F_r}, I_q\big)$, together with assumption , we have

$$S_{C\varepsilon} = \begin{pmatrix} H_G & 0 & 0 \\ 0 & H_{F_r} & 0 \\ 0 & 0 & I_q \end{pmatrix} \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} z_t \varepsilon_{t+h} \xrightarrow{d} \mathcal{N}\big(0, \Phi_0 \Omega \Phi_0'\big),$$

where $\Omega = \lim_{T\to\infty} \text{Var}\big(\frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} z_t \varepsilon_{t+h}\big)$.

For $\Delta_{C\varepsilon}$, it follows immediately from (a) and (b) Lemma 1 that, when , $\Delta_{C\varepsilon} \xrightarrow{p} 0$.

For Δ_{zG}, we have

$$\Delta_{zG} = \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} H_G G_t \\ H_{F_r} F_{r,t} \\ W_t \end{pmatrix} \big(\tilde{G}_t - H_G G_t\big)' \big(H_G^{-1}\big)' \alpha + \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} \tilde{G}_t - H_G G_t \\ \tilde{F}_{r,t} - H_{F_r} F_{r,t} \\ 0 \end{pmatrix} \big(\tilde{G}_t - H_G G_t\big)' \big(H_G^{-1}\big)' \alpha.$$

Then, by (d), (h), (j), (c) and (g) in Lemma 1, when , we have that $\Delta_{zG} \xrightarrow{p} 0$.

For Δ_{zF}, we have

$$\Delta_{zF} = \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} H_G G_t \\ H_{F_r} F_{r,t} \\ W_t \end{pmatrix} \big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' \big(H_{F_r}^{-1}\big)' \beta + \frac{1}{\sqrt{T}} \sum_{t=h+1}^{T} \begin{pmatrix} \tilde{G}_t - H_G G_t \\ \tilde{F}_{r,t} - H_{F_r} F_{r,t} \\ 0 \end{pmatrix} \big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' \big(H_{F_r}^{-1}\big)' \beta$$

Then, by (i), (f), (k), (c) and (g) in Lemma 1, when, we have that $\Delta_{zF} \xrightarrow{p} 0$.

For $S_{\hat{z}\hat{z}'}$, we have

$$\begin{aligned} S_{\hat{z}\hat{z}'} &= \frac{1}{T} \sum_{t=h+1}^{T} \begin{pmatrix} \tilde{G}_t - H_G G_t + H_G G_t \\ \tilde{F}_{r,t} - H_{F_r} F_{r,t} + H_{F_r} F_{r,t} \\ W_t \end{pmatrix} \Big(\tilde{G}_t - H_G G_t + H_G G_t \quad \tilde{F}_{r,t} - H_{F_r} F_{r,t} + H_{F_r} F_{r,t} \quad W_t \Big) \\ &= \begin{pmatrix} H_G & 0 & 0 \\ 0 & H_{F_r} & 0 \\ 0 & 0 & I_q \end{pmatrix} \Big(\frac{1}{T} \sum_{t=h+1}^{T} z_t z_t' \Big) \begin{pmatrix} H_G' & 0 & 0 \\ 0 & H_{F_r}' & 0 \\ 0 & 0 & I_q \end{pmatrix} + o_p(1) \\ &= \Phi_0 \Sigma_{zz'} \Phi_0' + o_p(1), \end{aligned}$$

where $\Sigma_{zz} = \operatorname{plim} \Big(\frac{1}{T} \sum_{t=h+1}^{T} z_t z_t' \Big)$ and the second equality follows from (c)-(k) in Lemma 1.

Then, combining the analyses above, back to (5), we have, if

$$\sqrt{T}\,\big(\hat{\theta}_T - \theta\big) \xrightarrow{d} \mathcal{N}\big(0, \Sigma_\theta\big),$$

where $\Sigma_\theta = \Phi_0'^{-1}\Sigma_{zz}^{-1}\Omega\Sigma_{zz}^{-1}\Phi_0^{-1}$.

Lemma 1. *Let Assumptions hold .Then,*

(a) $\frac{1}{T}\sum_{t=h+1}^{T}\big(\tilde{G}_t - H_G G_t\big)\varepsilon_{t+h} = O_p\big(\big)$;

(b) $\frac{1}{T}\sum_{t=h+1}^{T}\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)\varepsilon_{t+h} = O_p\big(\big)$;

(c) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T}\big(\tilde{G}_t - H_G G_t\big)\big(\tilde{G}_t - H_G G_t\big)' = O_p\big(\big)$;

(d) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} H_G G_t\big(\tilde{G}_t - H_G G_t\big)' = O_p\big(\big)$;

(e) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T}\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' = O_p\big(\big)$;

(f) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} H_{F_r} F_{r,t}\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' = O_p\big(\big)$;

(g) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T}\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)\big(\tilde{G}_t - H_G G_t\big)' = O_p\big(\big)$;

(h) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} H_{F_r} F_{r,t}\big(\tilde{G}_t - H_G G_t\big)' = O_p\big(\big)$;

(i) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} H_G G_t\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' = O_p\big(\big)$;

(j) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} W_t\big(\tilde{G}_t - H_G G_t\big)' = O_p\big(\big)$;

(k) $\frac{1}{\sqrt{T}}\sum_{t=h+1}^{T} W_t\big(\tilde{F}_{r,t} - H_{F_r} F_{r,t}\big)' = O_p\big(\big)$;

The second objective of the paper is to study the asymptotic properties of the forecasts:

$$\hat{y}_{T+h|T} = \hat{z}_T'\hat{\theta}_T. \tag{6}$$

By (4), we have

$$y_{T+h} = \hat{z}_T'\theta + \varepsilon_{r,T+h} + \alpha' H_G^{-1}(H_G G_T - \tilde{G}_T) + \beta' H_{F_r}^{-1}(H_{F_r} F_{r,T} - \tilde{F}_{r,T}).$$

Then, the forecast error is given by

$$\hat{y}_{T+h|T} - y_{T+h} = \hat{z}_T'\big(\hat{\theta}_T - \theta\big) + \alpha' H_G^{-1}(\tilde{G}_T - H_G G_T) + \beta' H_{F_r}^{-1}(\tilde{F}_{r,T} - H_{F_r} F_{r,T}). \tag{7}$$

We need to establish the following CLT:

$$\frac{\hat{y}_{T+h|T} - y_{T+h}}{\sqrt{\operatorname{var}(\hat{y}_{T+h|T})}} \xrightarrow{d} \mathcal{N}(0,1), \tag{8}$$

which can be used to construct the prediction intervals for the forecast. Since we have established the CLT for the parameter estimates, what remains are the inference results for the estimated factors. We summarize these results in the following lemma.

Lemma 2. *Let Assumptions hold .Then,*

(a)

$$?(\tilde{G}_T - H_G G_T) \xrightarrow{d} \mathcal{N}(0, ?)$$

(b)

$$?(\tilde{F}_{r,T} - H_{F_r} F_{r,T}) \xrightarrow{d} \mathcal{N}(0, ?)$$

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